DERIVATION OF ADAM- BASHFORTH EXPLICIT SCHEME USING CHEBYCHEV POLYNOMIAL AS BASIS FUNCTION

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Abstract: This research presents the derivation of linear multistep schemes for solving initial value problems, utilizing Chebyshev polynomials as basis functions. The employment of Chebyshev polynomials ensures an even distribution of error within the interval [-1,1], thereby improving the numerical accuracy and stability of the derived schemes

Keywords: Adams-Bashforth, Basis function, Chebyshev polynomial, Explicit schemes.

Introduction

The theory of Differential equation said Sophus lie is the most important branch of modern Mathematics. The subject may consider occupying a central position from which different lines of development extend in many directions. The study of differential equations began very soon after the invention of the differential and integral calculus to which it forms a natural sequel. Newton in 1676 solved a differential equation by the use of an infinite series only eleven years after his discovery of the fluxional form of the differential calculus in 1665. But results were not published until 1693, the same year in which a differential equation occurred for the first time in the work of Leibnitz (whose account of differential calculus was published in 1684. In the next few years progress was rapid, in 1694-97 John Bernoulli explained the method of "separating the variables" and he showed how to reduce a homogenous differential equation of the first order to one in which the variables were separable. He applied those method to problems on orthogonal trajectories. And his brother Jacob also known as James (after whom "Bernoulli's Equation" named) succeeded in reducing a large number of different equations to forms they could solve. Integrating factor was probably introduced by Euler (1734) and independently of him by Fontaine and Clairaut though also attributed to Leibnitz. Just to mention a few.

Solving differential equations of second order or higher were due to Euler D'Alembert dealt with the case when the auxillary equation had roots. Particular solutions were not given until about a hundred years after by Lobato (1837) and Boole (1859). Partial Differential Equations (PDE) was first given the form of a vibrating string of second order was discussed by Euler D' Alembert (1747). Lagrange completed the solution in 1772 to 1785. Alot of discovery and solution had been investigated ever since.

Definitions

Differential equations are mathematical equations that involve differential coefficients, such as equations 1-5. In general, a differential equation is an equation that relates an unknown function to one or more of its derivatives, often describing how the function changes over time or space.

$$\frac{d^2y}{dx^2} = -p^2y \qquad (1)$$

$$2\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 10y = e^{-3x}\sin 5x \qquad (2)$$

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} = 3e^{-3x}\frac{d^2y}{dx^2} \qquad (3)$$

$$\frac{dy}{dx} = \frac{x}{y^{\frac{1}{3}}\left(1+x^{\frac{1}{3}}\right)} \qquad (4)$$

$$\frac{\partial^3 y}{\partial x^3} = a\frac{\partial^2 y}{\partial x^2} \qquad (5)$$

Type

Differential equations can be broadly classified into two categories:

1. Ordinary Differential Equations (ODEs): These equations involve only one independent variable and are typically denoted as equations (1), (2), (3), and (4). ODEs describe how a function changes over a single variable.
2. Partial Differential Equations (PDEs): These equations involve two or more independent variables and include partial differential coefficients with respect to each variable, as seen in equation (5). PDEs describe how a function changes over multiple variables.

Order

The order of a differential equation is determined by the highest-order derivative present in the equation. For instance:

- Equation (1) and (3) is of the second order, as it involves a second-order differential coefficient.
- Equation (4) is of the first order, as it only involves a first-order differential coefficient.
- Equation (2) and (5) is of the third order, indicating that it involves a third-order differential coefficient.

In general, the order of a differential equation is defined by the highest-order derivative that appears in the equation.

Degree

The degree of a differential equation refers to the highest power or exponent of the highest-order differential coefficient, once the equation has been rationalized and integrated with respect to the differential coefficients. For example:

- Equations (1), (2), (4), and (5) are all of the first degree.

Equation (3) is not initially in rational form; however, after squaring to rationalize, it can be determined to be of a specific degree

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{3}}=9\left(\frac{d^2y}{dx^2}\right)^2 \qquad (6)$$

We then see that it is of the second degree as $\frac{d^2y}{dx^2}$ occurs squared.

Theoretical Underpinning

This section provides an overview of methods for solving first-order ordinary differential equations (ODEs). There are two primary approaches to solving differential equations the Analytic Method and the Numerical Methods, which will be discussed briefly:

Analytic Method

These methods involve finding an exact, closed-form solution to the differential equation using mathematical techniques such as separation of variables, integration factors, and undetermined coefficients.

In solving differential equations analytically, the follow methods can be adopted.

Variable Separable Method

A variable separable equation is a first order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y

It can be written in the form

$$\frac{dy}{dx}=f(x)g(x) \qquad (7)$$

The name separable comes from the fact that the expression on the right side can be "separated" into a function X and a function of y . Equivalently if $f(y)\neq 0$ we could write

$$\frac{dy}{dx}=\frac{g_{(x)}}{f_{(y)}} \qquad (8)$$

Where $h_{(y)}=\frac{1}{f_{(y)}}$

To solve (8) we rewrite it in the form $h_y dy=g_{(x)}dx$

So that all y 's are on one side of the equation and all X 's are on the other side. Then we integrate both sides of the equation. Thus, $\int h_{(x)}dy=\int g_{(x)}dx$

Example 1: Solve $\frac{dx}{x}=\tan y\, dy$

The R.H.S involves X only and the R.H.S only so the variables are separate.

Integrating, we have.

$\log x=-\log\cos y+c \log x\cos y=c$

$x\cos y=e^c \text{say} e^c=a$

$x\cos y=a$

Example 2: Solve $\frac{dy}{dx}=2xy$

We can separate the variable thus

Multiplying by dx and divide by y,we get

$\text{Log}\, x=x^2+c$

$y=ae^{x^2}$

Where c is an arbitrary constant

Exact Equations

Here we consider the expression

$$Mdx+Ndy=0 \qquad (9)$$

This is an exact differential equation if $M(x,y)$ and $N(x,y)$ are continuous differential functions for which the follow relationship is fulfilled.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \qquad (10)$$

$\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ are continuous in some region.

Integrating exact differential equations, we shall prove that the left side of equation (9) is an exact different equation then the condition (10) is fulfilled and conversely. If condition (10) is fulfilled the left side of equation (9) is exact differential of some function $U(x,y)$that is equation (9) is an equation of the form $dU(x,y)=0$ and consequently its complete integral is

$U(x,y)=c$

Let us first assume that the left side of (9) is an exact differential of some function $U(x,y)\ ie\ M(x,y)dx+N(x,y)dy=dU=\frac{du}{dx}dx+\frac{du}{dy}dy$, $N=\frac{du}{dx}$ $M\frac{dy}{dx}$

Differentiating, the first relation with respect to y and the second with respect to x we obtain.

$$\frac{\partial M}{\partial y}=\frac{\partial^2 U}{\partial x\partial y}, \qquad \frac{\partial N}{\partial x}=\frac{\partial^2 U}{\partial x\partial y}$$

Assuming continuity of the second derivatives, we have

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial y}$$

From the relation $\frac{\partial y}{\partial x}=M(x,y)$

we find

$$U=\int M(x,y)dx+\emptyset(y)$$

we differentiate with respect to y and equate the result to $N(x,y)$

$$\frac{\partial U}{\partial y}=\int\frac{\partial M}{\partial y}dx+\emptyset(y)=N(x,y)$$

but since $\frac{\partial U}{\partial y}=\frac{\partial N}{\partial x}$

we can write

$$U=\int\frac{\partial M}{\partial x}dx+\emptyset^I_{(y)}=N$$

i.e $N(x,y)+\emptyset(y)=N(x,y)$

$$\emptyset(y)=\int N(x,y)dx+c_1$$

Thus $U=\int M(x,y)dy+N(x,y)dy+c_1$

Example 1: Here, we solve $ydx+xdy=0$

$d(yx)=0$

i.e $yx=c$

Example 2: Solve $\frac{2x}{y^3}dx+\frac{y^2-3x^2}{y^4}dy=0$

Here, $M=\frac{2x}{y^3}$, $N=\frac{y^2-3x^2}{y^4}$

$$\frac{\partial M}{\partial y}=\frac{6x}{y^3}\frac{\partial N}{\partial x}=\frac{-6x}{y^4}$$

Since, $\frac{\partial m}{\partial y}=\frac{\partial N}{\partial x}$

Here it is an exact differential equation

Then, $\frac{\partial U}{\partial x}=\frac{2x}{y^3}$

So,

$$U=\int\frac{2x}{y^3}+\emptyset(y)=\frac{x^2}{y^3}+\emptyset(y)$$

$$\frac{\partial u}{\partial y} = N = \frac{y^2 - 3y^2}{y^3}$$
$$= \frac{-3x^2}{y^4} + \emptyset(y) = \frac{y^2 - 3x^2}{y^4}$$
$$\emptyset^I_{(y)} = \frac{1}{y^2}$$
$$\emptyset(y) = \frac{-1}{y} + c_1$$
$$U(x,y) = \frac{x^2}{y^3} - \frac{1}{y} + c$$

Homogenous method by substituting

$y = vx$ (11)

in this case we make the substitution of equation (11) and differentiate with respect to x

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ (12)

Example 1: We solve $\frac{dy}{dx} = \frac{x+3y}{2x}$

We substituting equation (11) and (12) and the equation now becomes

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$
$$= \frac{1 + 3v}{2}$$
$$x\frac{dv}{dx} = \frac{1 + 3v}{2} - v$$

The equation is now expressed in terms of V and X this form we find that we can solve by separating the variables, thus

$$\int \frac{2}{1+v}dv = \int \frac{1}{x}dx$$
$$2\log(1 + v) = \log x + c$$
$$\log(1 + v)^2 = \log x + \log a$$

But $y = vx$ so $v = \frac{y}{x}$

$$\left(1 + \frac{y}{x}\right)^2 = Ax$$

Which gives

$$(x + y)^2 = Ax^3$$

Integrating factor

When we have an equation of the form

$$\frac{dy}{dx} + p_y = Q$$

Where P and Q are function of x, this type of an equation is a linear equation of the first order. To solve any equation of such we multiply both side by an integrating factor which ie of the form.

$$I.F = e^{\int pdx}$$

this converts the L.H.S into the derivative f a product

Example 1: Solve $\frac{dy}{dx} - y = x$

If we compare this with $\frac{dy}{dx} + p_y = Q$ we see that $p = -1$ and $Q = 1$

hence, the integrating factor is:

$$e^{-\int pdx} = e^{-x}$$

we multiply both sides by e^{-x}

$$e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x}$$

that is

$$\frac{d}{dx}(e^{-x}y) = xe^{-x}$$
$$ye^{-x} = \int xe^{-x}\, d$$

The R.H.S integral can now be determined by integration by parts

$$ye^{-x} = x(-e^{-x}) + \int e^{-x}dx$$
$$= -xe^{-x} - e^{-x} + c$$
$$= -x - 1 + ce^x$$
$$\therefore y = ce^x - x - 1$$

Bernoulli's equation

These are equation of the form

$$\frac{dy}{dx} + p_y = Q^n$$

Where P and Q are functions of x (or constant) dividing both sides by $y^n$to have.

$$y^n\frac{dy}{dx} + Py^{1-n} = Q$$

and letting $Z = y^{1-n}$

So that differentiating

$$\frac{dz}{dx} = (1 - n)y^n\frac{dy}{dx}$$

Put $Z = y^{1-n}$ *and so* $\frac{dz}{dx} = (1 - n)y^{-n}\frac{dy}{dx}$

If we now multiply (2.6) by (1-n) we shall convert the first term into $\frac{dz}{dx}$ that is

$$(1 - n)y^{-n}\frac{dy}{dx} + (1 - n)py^{1-n} = (1 - n)Q$$

Consequently, we have

$$\frac{dz}{dx} + P_{I^{(z)}} = Q_I$$

Where Q_I $and P_I$ are function of x

Example 1: Solve $\frac{dy}{dx} + \frac{1}{x}y = x$

Divide both sides by y^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x$$

Letting $Z = y^{1-n} = y^{-1}$

$$\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$$

Multiply through by –1 to have

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -x$$

So that $\frac{dz}{dx} - \frac{1}{x}z = -1$

Which is of the form $\frac{dz}{dx} + P_Z = Q$

So that the equation can be solved by the normal integrating factor method

$$I.F = \frac{1}{x}$$
$$Z\frac{1}{x} = -\int dx$$
$$Z = cx - x^2$$

so $y = (cx - x)^{-1}$

Numerical Methods

These methods involve approximating the solution to the differential equation using numerical algorithms, such as Euler's method, Runge-Kutta methods, and finite difference methods. Numerical methods are often used when analytical solutions are not possible or are difficult to obtain. Numerical methods play a crucial role in solving differential equations, particularly those whose analytical solutions are intractable or difficult to obtain. In fact, only a limited number of differential equations can be solved exactly in terms of familiar functions. For the majority of equations, approximate numerical methods must be employed.

Furthermore, even when analytical methods are applicable, they often provide approximations that are only accurate for a short range of x values. In such cases, numerical methods can provide more accurate and reliable solutions over a broader range of values. In such where a differential equation and known bound condition are given an approximate solution is often obtainable by the application of numerical methods. Starting with Picard's method of getting successive algebraic approximations. By putting values in these, we generally get excellent numerical results. Unfortunately, the method can be applied to a limited class of equation. Some various methods frequently used numerical method are methods such as Euler, Runge-Kutha, Adams-Bashforth and predictor-corrector method.

Euler's method

For a first order differential equation

$$y' = f(x, y) \quad (13)$$

with initial condition

$$y_{x_0} = y_0 \quad (14)$$

Euler's method is the simplest of Approximation techniques. A step length h is chosen and y is approximated at the point $x_n = x_0 + nh$. The approximation to $y(x_n)$being devoted by y_n

The direction field for the differential equation (13) consist at each point of the (x, y) plane, of the slope of the solution to (13) which passes through that point that is, a straight line for a step length h in the x-direction.

Alternatively, the solution of (13) passing through the point (x_n, y_n) has slope $f(x_n, y_n)$ at the point and so a first order, Taylor's expansion of this solution about x_n leads to the approximation. $y_{n+1} = y_n + hf(y_n x_n)$

Mid-Point Method

The midpoint method also known as the second order Runge- Kutta method, improver the Euler method by adding a mid-point in the step which increase the accuracy by one order.

To solve a first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

Given the initial condition

$y_{x_0} = y_0$ and pick the marching step h

$$k_1 = hf(x_n y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_I}{2}\right)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_I}{2}\right)$$

$$y_{n+1} = y_n + k_2 + 0(h^3)$$

Runge – Kutta Method

One of the most widely used fixed step length Runge-Kutta methhod is the classical Runge-Kutta or R4 method which uses a four-stage formula

To solve a first order ordinary differential equal

$$\frac{dy}{dx} = f(x, y)$$

Given the initial condition $y_{x_0} = y_0$ and given the marching step h

$$k_1 = h(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{k_I}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + 0(h^5)$$

Linear Multistep Method

Linear multistep methods are used in mathematics for the Numerical solution of ordinary differential equations. Linear multistep methods can be used to solve into value problems of the form.

$$y^I = f(x, y) y_{x_0} = y_0$$

Consider for example the problem

$$y^I = y y_{(0)} = y_0$$

The exact equation is

$$y_{(x)} = e^t$$

A simple numerical method is Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Euler's method is a one-step method. A simple multistep method is the two step Adams-Bashforth method which is

$$y_{n+2} = y_{n+1} + \frac{3}{2} hf(x_n, y_n) - \frac{1}{2} hf(x_n, y_n)$$

This method needs two values y_{n+1} and y_n to compute the next y_{n+2}level

Generally linear multistep method is a method of form

$$\sum_{s=0}^{n} \alpha_s y_{n+s} = h \sum_{s=0}^{n} \beta_s f_{n+s} \quad (15)$$

A numerical method is classified as explicit if $\beta_s = 0$, and implicit if $\beta_s \neq 0$, with the coefficients $\alpha_o \dots \alpha_{s-1}$ and $\beta_0 \dots \beta_s$ determining the specific method, where h denotes the step size and f represents the right-hand side of the differential equation.

Examples

The simplest of all Adams method is the explicit one step method which is just Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n)$$

The simplest Adams-Moultons method is also the one step method which is of the form

$$y_{n+1} = y_n + \frac{1}{2}[f(x_{n+1}, y_{n+1}) - f(x_n, y_n)]$$

The co-efficient are those of the trapezoidal rule so the one step Adams Moultons method is the implicit trapezoidal formula.

The two-step formulas are:

Adams – Bashforth $n = 2$

$$y_{n+1} = y_n + \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})]$$

Adams moulton $n = 2$

$$y_{n+1} = y_n + \frac{h}{12}[5f(x_{n+1}, y_{n+1}) + 8f(x_n, y_n) - f(x_{n-1}, y_{n-1})]$$

It is of great importance to note that implicit method entails a substantially greater computational effort than the explicit method. On the other hand, for a given step number k implicit methods can be made more accurate than explicit ones. So, an N step Adams Bashforth formula has global truncation error of order $0(h^n)$ where N-step Adams Moulton is of order $0(h^{n+1})$

One way of deriving the implicit formular in the backward Euler is to simply iterate it with the initial value of the iteration y_{n+1}

$$y_{n+1}^{m+1} = y_n + hf(x_{n+1}, y_{n+1}^{m}) \quad (16)$$

The predictor-corrector approach involves a two-step process, where an explicit method is first used to predict a value, and then an implicit method is used to correct and refine that value. This approach is particularly useful in numerical analysis. In predictor-corrector methods, an explicit formula (such as Adams-Bashforth) is used to predict the value of y_{n+1}. Then, an implicit formula (such as Adams-Moulton) of the same order is used to correct and improve the predicted value. The implicit formula typically requires one fewer step but utilizes more recent information, resulting in improved accuracy. The predictor-corrector pair typically consists of an Adams-Bashforth method and an Adams-Moulton method of the same order, working together to provide a more accurate solution. The simplest pair would be the two-step Adam Bashforth method.

$$y_{n+1} = y_n + \frac{h}{2}[3(x_n, y_n) - f(x_{n-1}, y_{n-1})] \quad (17)$$

as the predictor and the implicit trapezoidal rule as corrector. One step of this predictor- corrector method would thus consist of computing

$$y_{n+1}^{p} = y_n + \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] \quad (18)$$

and then

$$y_{n+1} = y_{n+1}^{c} + y_n + \frac{h}{2}\left[(x_{n+1} y_{n+1}^{p}) + f(x_n, y_n)\right] \quad (19)$$

Another way of deriving implicit formula is to use newton iteration

Order of Multistep Methods

Consider the linear difference operator £ defined by

$$\pounds\left(y_{(x)}: h\right) = \sum_{i=0}^{k}[\alpha_i y(x + jh) - h\beta_i y^{I}(x + jh)] \quad (20)$$

where $y(x)$ is an arbitrary function continuously differential [a,b], we can formally define the order of accuracy of the operator and of the associated linear multimethod without involving the solution of the initial value problem (20). Expanding the test function $y(x + jh)$ and its derivative$y^{I}(x + jh)$ as Taylor series about x and collecting terms in equation (20) gives.

$$\pounds\left(y_{(x)}: h\right) = c_o y_{(x)} + c_1 h y_{(x)}^{l} + \ldots + c_q h^h y_{(x)}^{q} \quad (21)$$

where c_q are constants

The difference operator (20) and the associated linear multistep method (21) are said to be of order p if

$c_0 = c_1 = . \; . \; . = c_p = 0$

and

$c_{p+1} \neq 0$

where

$$c_0 = \sum_{j=0}^{k} \alpha_j$$

$$c_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

$$c_q = \frac{1}{q!}\sum_{j=1}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j$$

Derivative of Some Linear Multistep Schemes

Chebyshev's polynomials

In this section, we shall present the derivative of some linear multistep schemes in solving initial value problem. We shall make use of Chebyshev 's polynomials as basis function the reason for the use of Chebyshev's polynomials is as a result of the even distribution of error in the range [-1,1].

The r^{th} degree Chebychev polynomials

$$T_{r(x)} = cos\,[R\,cos^{-1}\{\tfrac{2x-b-a}{b-a}\}] \cong \sum_{m=0}^{n} c_m^r\, x^m,$$

$a \leq x \leq b,$

Which satisfies the recurrence relation

$$T_{r+1^{(x)}} = 2\left[\frac{2x-b-a}{b-a}\right]T_{r^{(x)}} - T_{r-1^{(x)}} \quad r \geq 1$$

Where

$$T_{0^{(x)}} = 1\,,\, T_{r^{(x)}} = \frac{2x-b-a}{b-a}$$

The choice of Chebyshev polynomial as basis function in this project as a result, all monomials in $[a, b]$ has the least maximum magnitude of error.

In this chapter, we shall present the derivative of some classes of explicit methods which can be used as corrector when solving first order Ordinary Differential Equation with any initial value solvers.

Here, we consider the solution to the First Order initial value problem

$$y = f(x, y_{(x)}); x_k \leq x \leq x_{k+n} \quad (22)$$

$y_{(x_k)} = y_k$

In this case, we make use of the Chebyshev polynomial

$$y_{(x)} = \sum_{k=0}^{n} a_r\, T_{r^{(x)}}\,; x_{(x)} \leq x \leq x_{(x+n)} \quad (23)$$

Where $T_{r^{(x)}}$ is the Chebychev polynomial equation 3.2 can be re-expressed as

$$y_{(x)} = \sum_{r=0}^{n} T_r\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right)$$

A one step method

Here we consider the case of $n = 2$ in (24)

$$y_{(x)} = a_0 T_{r^{(x)}} + a_1 T_{1^{(x)}} + a_2 T_{2^{(x)}} \quad (25)$$

by the use of (25) and the equivalent value of

$$T_{r^{(x)}} y_{(x)} = a_0 + a_{1\left(\frac{x}{h}-k-1\right)} + a_{2\left(\frac{x}{h}-k-1\right)} \quad (26)$$

$$y'_{(x)} = \frac{2a_1}{h} + \frac{8a_{2\left(\frac{x}{h}-k-1\right)}}{h} \quad (27)$$

Collocating (26) at x_k, x_{k+1} and interpolating (27) atx_{k+1} gives the set of algebraic equation

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} Y_{k+1} \\ hf_k \\ hf_{k+1} \end{pmatrix} \quad (28)$$

Solving equation (28) this equation gives

$$a_0 = Y_{k+1} + \frac{h}{4} f_{k+1} - f_k \quad (29)$$

$$a_1 = hf_{k+1} \quad (30)$$

$$a_2 = \frac{h}{4}(f_{k+1} - f_k) \quad (31)$$

We Substituting equations a_0 , a_1 and a_2 into equation (26) to give the continuous scheme

$$y_k = y_{k+1} + \frac{hf_{k+1}}{4}\left\{2\frac{(x-x_k)^2}{h^2} - 2\right\} - \frac{hf_k}{4}\left\{2\frac{(x-x_k)^2}{h^2} - 4\frac{(x-x_k)}{h} + 4\right\} \quad (32)$$

Evaluating (32) at x_{k+1} ,we have the discrete scheme.

$$y_{k+2} = y_{k+1} + \frac{h}{2}(3f_{k+1} - f_k)$$

A two-step method

Here, we make use of (24) such that $n = 3$**,** that is

$$y_x = a_0 T_{0(x)} + a_1 T_{1(x)} + a_2 T_{2(x)} + a_3 T_{3(x)}$$

By the use of (25) and $n = 3$ the equivalent value of $T_{r(x)}$

$$y_{(x)} = a_0 + a_1\left(\frac{2x}{3h} - \frac{2k}{3} - 1\right) + a_2\left(2\left(\frac{2x}{3h} - \frac{2k}{3} - 1\right)^2 - 1\right) + a_3\left(4\left(\frac{2x}{3h} - \frac{2k}{3} - 1\right)^3 - 3\left(\frac{2k}{3h} - \frac{2k}{3} - 1\right)\right) \quad (33)$$

$$y'_x = \frac{2a_1}{h} + \frac{8a_2}{3h}\left(\frac{2x}{3h} - \frac{2k}{3} - 1\right) + \frac{2a_3}{h} \quad (34)$$

Collocating (33) at x_k, x_{k+1} , x_{k+2} and interpolating (34) at x_{k+2} gives the following set of algebraic equation

$$\begin{pmatrix} 27 & 9 & -21 & -23 \\ 0 & 2 & -8 & 18 \\ 0 & 6 & -8 & -10 \\ 0 & 6 & 8 & -10 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 27y_{k+2} \\ 2hf_k \\ 2hf_{k+1} \\ 2hf_{k+2} \end{pmatrix} \quad (36)$$

Solving these equation gives

$$a_0 = y_{k+2} + \frac{h}{192}(44f_{k+2} - 148f_{k+1} + 8f_k),$$

$$a_1 = \frac{h}{64}(63f_{k+2} + 18f_{k+1} + 15f_k)$$

$$a_2 = \frac{9h}{16}(f_{k+2} - f_{k+1})$$

$$a_3 = \frac{9h}{64}(f_{k+2} - 2f_{k+1} + f_k)$$

Substituting these into (34) gives the continuous scheme

$$y_x = y_{k+2} + \frac{hf_{k+2}}{192}\left\{108\left(\frac{x-x_k}{3h}\right)^3 + 216\left(\frac{x-x_k}{3h}\right)^2 + 108\left(\frac{x-x_k}{3h}\right) - 64\right\} - \frac{hf_{k+1}}{192}\left\{216\left(\frac{x-x_k}{3h}\right)^3 + 216\left(\frac{x-x_k}{3h}\right)^2 + 216\left(\frac{x-x_k}{3h}\right) - 40\right\} + \frac{hf_k}{192}\left\{108\left(\frac{x-x_k}{3h}\right)^3 - 36\left(\frac{x-x_k}{3h}\right) + 8\right\} \quad (37)$$

Evaluating at x_{k+3}, we have the discrete scheme

$$y_{k+3} = y_{k+2} + \frac{h}{12}(23f_{k+2} - 16f_{k+1} + 5f_k)$$

A three-step method

Here we consider the case where $n = 4$ in (24), that is

$$y_k = a_0 T_{0(x)} + a_1 T_{1(x)} + a_2 T_{2(x)} + a_3 T_{3(x)} + a_4 T_{4(x)}$$

by the use of (25) and the equivalent values of $T_{r(x)}$

$$y_{(x)} = a_0 + a_1\left(\frac{x}{2h} - \frac{k}{2} - 1\right) + a_2\left(2\left(\frac{x}{2h} - \frac{k}{2} - 1\right)^2 - 1\right)$$
$$+ a_3\left(4\left(\frac{x}{2h} - \frac{k}{2} - 1\right)^3 - 3\left(\frac{x}{2h} - \frac{k}{2} - 1\right)\right)$$
$$+ a_4\left(8\left(\frac{x}{2h} - \frac{k}{2} - 1\right)^4 - 8\left(\frac{x}{2h} - \frac{k}{2} - 1\right)^2 + 1\right) \quad (38)$$

$$y'_x = \frac{a_1}{2h} + \frac{2a_2}{h}\left(\frac{x}{2h} - \frac{k}{2} - 1\right) + \frac{3a_3}{2h}\left(4\left(\frac{x}{2h} - \frac{k}{2} - 1\right)^2 - 1\right)$$
$$+ \frac{8a_4}{h}\left(2\left(\frac{x}{2h} - \frac{k}{2} - 1\right)^3 - \left(\frac{x}{2h} - \frac{k}{2} - 1\right)\right) \quad (39)$$

Collocating (38) at x_k, x_{k+1}, x_{k+2} *and* x_{k+3} and interpolating (39) at x_{k+3} gives the sets of algebraic equation

$$\begin{pmatrix} 0 & 1 & -1 & -2 & -1 \\ 0 & 1 & -4 & 9 & -16 \\ 0 & 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 2y_{k+3} \\ 2hf_k \\ 2hf_{k+1} \\ 2hf_{k+2} \\ 2hf_{k+3} \end{pmatrix}$$

Solving these equation gives

$$a_0 = y_{k+3} + \frac{h}{24}\{3f_{k+3} - 19f_{k+2} - 7f_{k+1} + 19f_k\}$$
$$a_1 = h\{f_{k+3} + f_{k+1}\}$$
$$a_2 = \frac{h}{6}\{4f_{k+3} - 3f_{k+2} - f_k\}$$
$$a_3 = \frac{h}{3}\{f_{k+3} - 2f_{k+2} + f_{k+1}\}$$
$$a_4 = \frac{h}{3}\{f_{k+3} - f_{k+2} + f_{k+1}\}$$

Substituting these into (38) gives the continuous scheme

$$y_x = y_{k+3} + \frac{hf_{k+3}}{24}\left\{16\left(\frac{x-x_k}{4h}\right)^4 + 32\left(\frac{x-x_k}{4h}\right)^3 + 16\left(\frac{x-x_k}{4h}\right) - 9\right\} - \frac{hf_{k+3}}{24}\left\{48\left(\frac{x-x_k}{4h}\right)^4 + 64\left(\frac{x-x_k}{4h}\right)^3 - 24\left(\frac{x-x_k}{4h}\right)^2 - 48\left(\frac{x-x_k}{4h}\right) + 19\right\} + \frac{hf_{k+1}}{24}\left\{48\left(\frac{x-x_k}{4h}\right)^4 + 32\left(\frac{x-x_k}{4h}\right)^3 - 48\left(\frac{x-x_k}{4h}\right)^2 + 5\right\} - \frac{hf_k}{24}\left\{16\left(\frac{x-x_k}{4h}\right)^4 - 8\left(\frac{x-x_k}{4h}\right)^2 + 1\right\} \quad (40)$$

Evaluate (40) at x_{k+4} we have the discrete scheme

$$y_{k+4} = y_{k+3} + \frac{h}{24}\{55f_{k+3} - 59f_{k+2} + 37f_{k+1} - 9f_k\}$$

Convergence

In this section we shall look at some basic properties of multistep methods

i. Consistence
ii. Zero stability

For any linear multistep to converge it must be consistence and zero stable.

Definition

1. A multistep is consistence if it has order p≥ 1
2. A linear multistep is zero stable if no roots of the first characteristic polynomial $p(\varepsilon)$ has modulus greater than one and if every root with modulus one is simple. Here we shall determine the ordeer of our methods.
3. A linear multistep is of order p if for $c_0 = c_1 = \cdot\cdot\cdot = c_p = 0$ but $c_{p+1} \neq 0$
   Where $c_0 = \sum_{j=0}^{k} \alpha_j$
   $$c_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$
   $$c_q = \frac{1}{q!}\sum_{j=1}^{k} J^q\alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j$$

We find the orders as follows

$$y_{k+2} = y_{k+1} + \frac{h}{2}\{3f_{k+1} - f_k\}$$

$$c_0 = \sum_{j=0}^{2} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2$$

$= 0-1 + 1= 0$

$$c_1 = \sum_{j=1}^{2} j\,\alpha_j - \sum_{j=0}^{1} \beta_j$$

$$= \alpha_1 + 2\alpha_2 - \beta_0 + \beta_1$$

$$= -1 + 2 + \frac{1}{2} - \frac{3}{2} = 0$$

$$c_2 = \frac{1}{2!}\sum_{j=1}^{2} j\alpha_j - \sum_{j=1}^{1} j^1\beta_j$$

$$= \frac{1}{2}(1^2\alpha_1 - 2^2\alpha_2) - 1\beta_1$$

$$= \frac{3}{2} + \frac{3}{2} = 0$$

$$c_3 = \frac{1}{3!}\left(\sum_{j=1}^{2} \alpha_j\right) - \frac{1}{(3-1)!}\sum_{j=1}^{1} \beta_j$$

$$= \frac{1}{6}(1^3\alpha_1 + 2^3\alpha_2) - \frac{1}{2}(1^2\beta_1)$$

$$= \frac{7}{6} - \frac{3}{4} = \frac{5}{12}$$

This is consistent of order 2 and the error constant is $\frac{5}{12}$

$$y_{k+3} = 12y_{k+2} + \frac{h}{12}(23f_{k+2} - 16f_{k+1} + 5f_k)$$

$$c_0 = \sum_{j=0}^{3} \alpha_j$$

$$= 0 + 0 - 12 - 12 = 0$$

$$c_1 = \sum_{j=1}^{3} j\alpha_j - \sum_{j-0}^{2} \beta_j = 0$$

$$c_2 = \frac{1}{2!}\sum_{j=1}^{4} j^2\alpha_j - \sum_{j=1}^{2} \beta_j$$

$$= 30 - 30 = 0$$

$$c_3 = \frac{1}{4!}\sum_{j=1}^{4} j^4\alpha_j - \frac{1}{(3)!}\sum_{j=1}^{2} j^3\beta_j$$

$$= 38 - 38 = 0$$

$$c_4 = \frac{1}{4!}\sum_{j=1}^{4} j^4\alpha_j - \frac{1}{3!}\sum_{j=1}^{2} j^3\beta_j$$

$$= \frac{9}{2}$$

This is consistent, of order 3 and the error constant is $\frac{9}{2}$

$$y_{k+4} = y_{k+3} + h(55f_{k+3} - 59f_{k+2} + 37f_{k+1} - 9f_k)$$

$c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = \frac{251}{30}$

This is consistent, of order 4 and the error constant is $\frac{251}{30}$

Zero Stability

In this section we shall take a look at zero stability of linear multistep scheme

Definition: A linear multistep method is zero stable if no root of the first characteristic polynomial $P(\varepsilon)$ has modulus greater than one and if every root with modulus one is simple

To calculate the zero stability of the methods derived in this chapter, we shall consider the roots of the first characteristic polynomial $P(\varepsilon)$ Of a linear multistep method

$$\alpha y_n + \alpha_1 y_{n+1} + \cdots + \alpha_n y_{n+k} = h\sum_{j=0}^{k} \beta_j$$

The root of the first characteristic polynomial is given as

$$\alpha_k r^k + \alpha_{k-1} r^{n-1} + \cdots + \alpha_1 r + \alpha_0 = 0$$

Applying this to the scheme

$$y_{k+2} - y_{k+1} = \frac{h}{3}(3f_{k+1} + f_k)$$

Here $r = 0\ or\ 1$

This is zero stable

$$12y_{k+3} - 12y_{k+2} = h(23f_{k+2} - 16f_{k+1} + 5f_k)$$

here $r = 0\ or\ 1$

so, this is zero stable

$$24f_{k+4} - 24f_{k+3} = h(55f_{k+3} - 59f_{k+2} + 37f_{k+1} - 9f_k)$$

here $r = 0\ or\ 1$

Therefore, this is zero stable.

Table 1: Error Constant

Method	Order	Error Constant
A one step method	2	$\frac{5}{12}$
A two step method	3	$\frac{9}{2}$
A three step method	4	$\frac{251}{30}$

Conclusion

In this presentation we looked at two methods of solving First Order Ordinary Differential Equation.

(1) Analytical
(2) Numerical

In solving the differential equations analytically, the following methods can be adopted:

Variable Separable, Exact Equations, Homogeneous method, Integrating Factor method to mention just a few

The numerical methods cannot be over looked, because it affords us the opportunity of solving differential equations whose analytical solution is intractable. So, some various methods frequently used are: Euler's Method, Runge-Kutta, Mid-Point, Linear multistep Methods which is always

We have derived both continuous and discrete Adams Bashforth Linear Multistep Explicit Scheme for solving First Order Ordinary Differential Equations. In this derivation, we made use of Chebyshev Polynomial as the basis function. The reason being as a result of even distribution of error in the range $[-1,1]$.

The Linear Multistep Scheme yields Adams-Bashforth Explicit method of the One step, two step, and the Three Step Method at grid point which serve as the predictor to Adams Moulton of the same order to correct or improve that value.

Convergence is a minimal property which any acceptable linear multistep method must possess. Qualitatively speaking, consistency controls the magnitude of the local truncation errors committed at each stage of the calculation while zero stability controls the manner in which this error is propagated as the calculation proceeds, both are essential if convergence is to be achieved.

The explicit Adams-Bashforth Scheme derived meets the necessary and sufficient condition to three convergent which is both consistent and zero stable. So, we recommend the Adams-Bashforth Multistep Explicit Scheme as predictor for Solving First Order Ordinary Differential Equation.

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Conflict of Interest

Authors declare no conflict of interest.

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